DO ARBITRAGE FREE PRICES COME FROM UTILITY MAXIMIZATION?

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SHOULD I BUY OR SELL?

ARBITRAGE FREE PRICES

ALWAYS BUY | IT DEPENDS | ALWAYS SELL

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MARKET

ARBITRAGE FREE PRICES

ALWAYS BUY | IT DEPENDS | ALWAYS SELL

MARKET + AGENT

MARGINAL PRICES

BUY | DO NOTHING | SELL

MARGINAL PRICES

Agent

- $u(x, q)$ maximal expected utility achievable
- x initial cash wealth
- q initial number of cont. claims

Marginal Prices

Intuitive definition

p is a marginal price for the agent with utility u and initial endowment (x, q) if his optimal demand of cont. claims at price p is zero.

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Marginal Prices

Definition of $\mathcal{MP}(x, q; u)$

p is a marginal price at (x, q) relative to u if

$$u(x - pq', q + q') \leq u(x, q) \quad \text{for all } q' \in \mathbb{R}^n,$$

QUESTIONS

1. Are marginal prices always arbitrage free ?

$$\mathcal{MP}(x, q; u) \subseteq \mathcal{AFP} \quad ?$$

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$$\mathcal{MP}(x, q; u) \subseteq \mathcal{AFP} \quad ?$$

KARATZAS AND KOU (1996)

2. Do all arbitrage free prices come from utility maximization?

$$\bigcup \mathcal{MP}(x, q; u) \supseteq \mathcal{AFP} \quad ?$$

Union over what ?

THE MARKET

Liquid frictionless market

- Bank account, with no interest
- Stocks: semimartingale S, admitting ELMMs
- (Almost) no constraints on strategy H

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Illiquid contingent claims

- $f(\omega) \in \mathbb{R}^n$ random payoff
- $|f| \leq c + \int_0^T H dS$ for some c, H
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Definition of $\mathcal{AFP}$

p is an arbitrage free price if

$$q'(f - p) + \int_0^T HdS \geq 0 \quad \text{implies} \quad q'(f - p) + \int_0^T HdS = 0$$

UTILITY, MARGINAL PRICES

Agent

- Maximal expected utility

$$u(x,q) := \sup_H \mathbb{E}[U(x + qf + \int_0^T H dS)]$$

- $U : (0,\infty) \to \mathbb{R}$ Utility: strictly concave, increasing, differentiable, Inada conditions

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i.e. if (x,q) maximizes u over $\{(x - pq', q + q') : q' \in \mathbb{R}^n\} =: A$

Setting as in HUGONNIER AND KRAMKOV (2004)

MAIN THEOREM

Theorem

If $\sup_x(u(x,0) - xy) < \infty$ for all $y > 0$ then

$$\bigcup_{(x,q)\in\{u>-\infty\}} \mathcal{MP}(x,q;u) = \mathcal{AFP}$$

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- $u(x,0) = u(x)$ as in Kramkov and Schachermayer (1999)
- Any U is enough to reconstruct $\mathcal{AFP}$
- Enough to consider small (x,q)
- Always we need (x,q) close to $\partial\{u > -\infty\}$
- In general we need $(x,q) \in \partial\{u > -\infty\}$

BOUNDARY POINTS ARE ILL-BEHAVED

Technical reasons

The multi-function $\mathcal{MP} : int\{u > -\infty\} \rightarrow \mathbb{R}^n$

$$(x, q) \mapsto \mathcal{MP}(x, q; u)$$

has compact, non-empty values and is upper-hemicontinuous

...NONE of this is true on the boundary !

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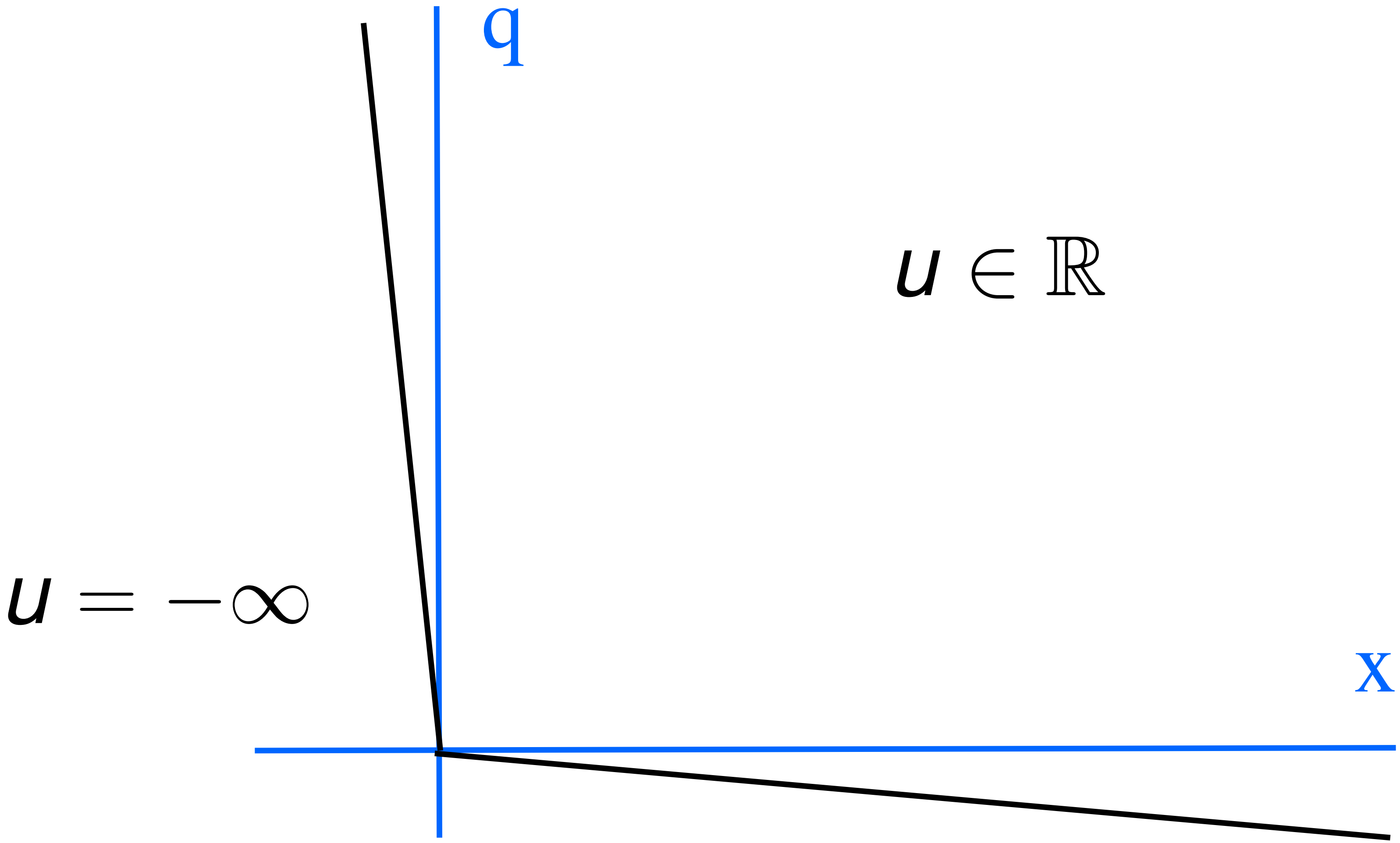
Economic reasons

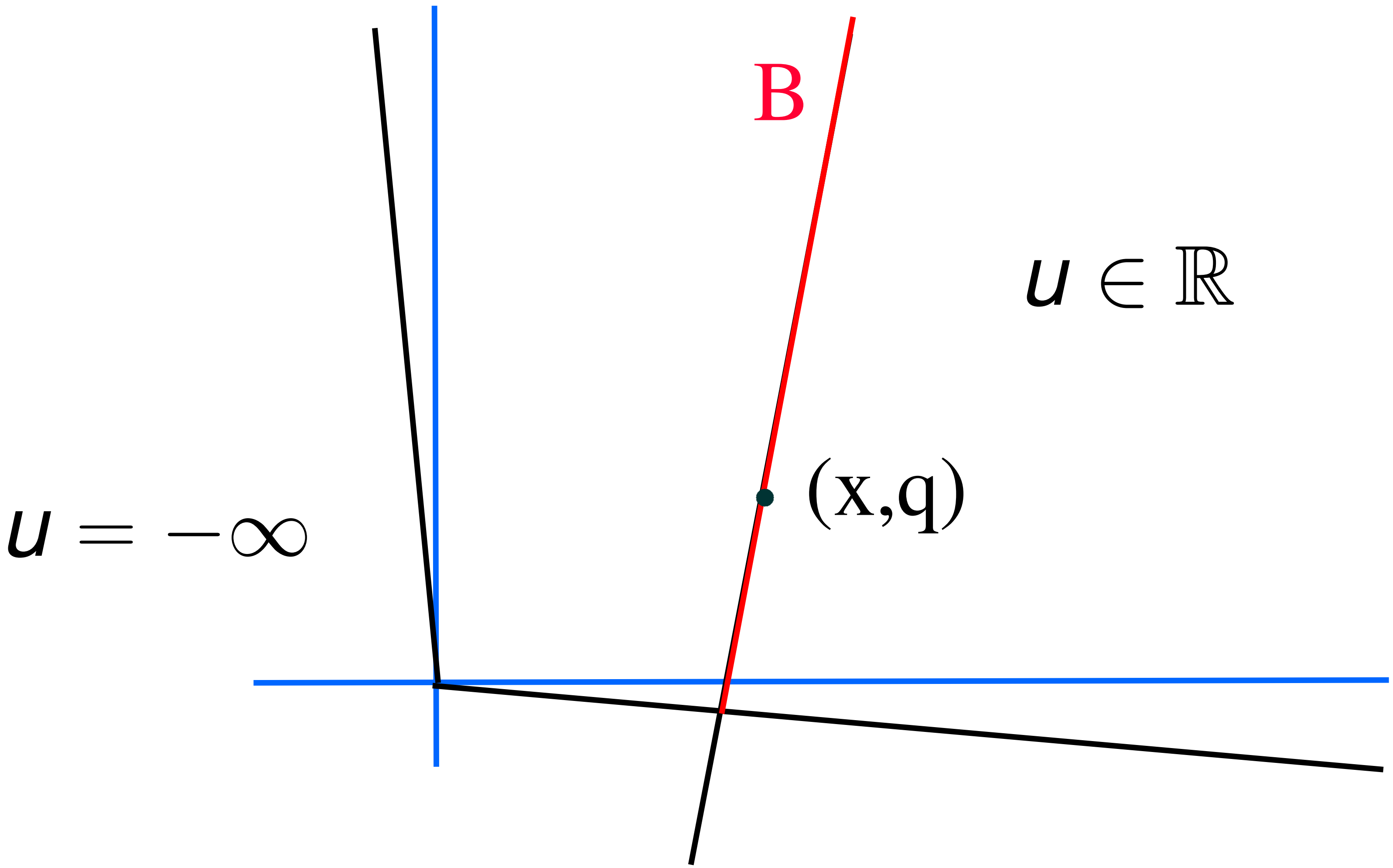
Theorem

If $p_0 \in \mathcal{P}(x, q)$ for some non-zero $(x, q) \in \partial\{u > -\infty\}$,

then $\exists p \in \mathbb{R}^n \setminus \mathcal{AFP}$ such that $[p_0, p) \subseteq \mathcal{MP}(x, q; u)$

DOMAIN OF UTILITY u





$$A := \{(x - pq', q + q') : q' \in \mathbb{R}^n\}$$

$p \in \mathcal{MP}(x, q; u)$ if (x, q) is maximizer of u on B

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SKETCH OF PROOF

New geometric characterization of $\mathcal{AFP}$

The following are equivalent:

1. $p \in \mathcal{AFP}$
2. B is bounded
3. If $(x', q') \in cl\{u > -\infty\}$ satisfies $x' + q'p = 0$ then $(x', q') = (0, 0)$
4. There exists an ELMM $\mathbb{Q}$ such that $p = \mathbb{E}^{\mathbb{Q}}[f]$ etc.

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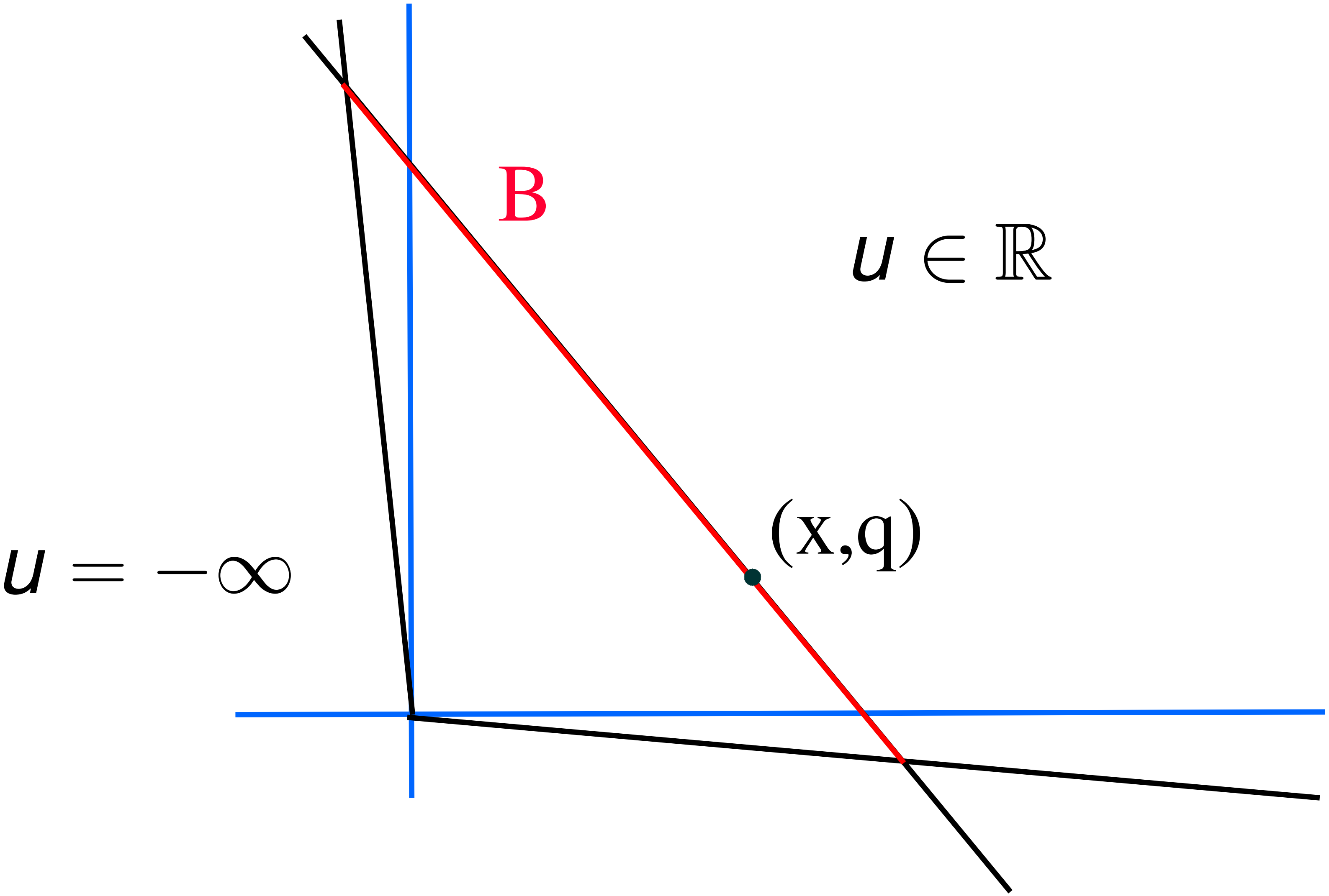
$(x', q') \in cl\{u > -\infty\}$, taking $(x', q') = (-q'p, q')$ as in item (4)

gives $u(x, q) < u(x - q'p, q + q')$

P ARBITRAGE FREE PRICE



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u is upper semi-continuous

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- Take $(x^k, q^k) \to (x, q), H^k$ s.t. $W^k := x^k + q^k f + (H^k \cdot S)_T$ satisfies $\mathbb{E}[U(W^k)] = u(x_k, q_k) \to s \in \mathbb{R}$

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- Use duality theory to show that $\exists H$ s.t. $V \leq W := x + qf + (H \cdot S)_T$, so $\mathbb{E}[U(V)] \leq u(x, q)$

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- By Jensen inequality $\mathbb{E}[U(V^k)] \geq inf_{n \geq k} \mathbb{E}[U(W^n)]$
- Show that $U(V^k)^+$ is uniformly integrable, so by Fatou $\overline{\lim}_k \mathbb{E}[U(V^k)] \leq \mathbb{E}[U(V)]$, so $\overline{\lim}_k u(x_k, q_k) \leq u(x, q)$

SUMMARY

Arbitrage free prices come from utility maximization

$$\bigcup_{(x,q)\in\{u>-\infty\}} \mathcal{MP}(x,q;u) = \mathcal{AFP}$$

In general we need also $(x,q) \in \partial\{u > -\infty\}$

The corresponding $p_0 \in \mathcal{MP}(x,q)$ are quirky

$\exists p \in \mathbb{R}^n \setminus \mathcal{AFP}$ such that $[p_0, p) \subseteq \mathcal{MP}(x,q)$